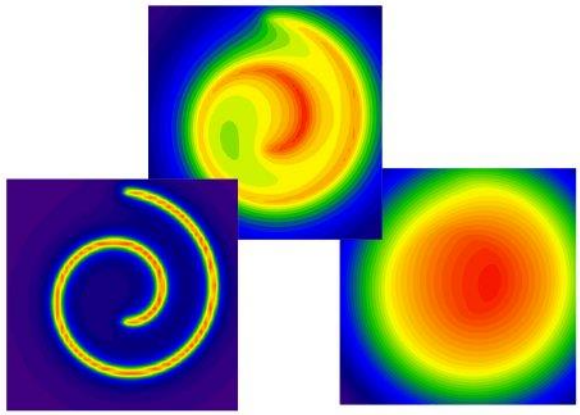
OECD SHORT COURSE Nov. 24-28 2025
ANALYTICAL BENCHMARKS: CASE STUDIES IN NEUTRON TRANSPORT THEORY*

Monday 24	9:00- 10:15	Introduction
	10:15-10:30	Coffee break
	10:30-12:00	Where it all begins
	12:00-13:30	Lunch break
	13:30-15:00	Derivation of Boltzmann Eqn
	15:00-15:15	Coffee break
	15:15-17:00	Derivation of Neutron Transport Equation
	17:00-17:30	Selected Derivations
Tuesday 25	9:00- 10:15	Integral Transport Equation
	10:15-10:30	Coffee break
	10:30-12:00	Solutions Without Scattering
	12:00-13:30	Lunch break
	13:30-15:00	1D Integral Transport and Solutions
	15:00-15:15	Coffee break
	15:15-17:00	1D Plane Geometry Benchmark
	17:00-17:30	Introduction of OECD Analytical Benchmark Library
Wednesday 26	9:00- 10:15	Neutron Slowing Down
	10:15-10:30	Coffee break
	10:30-12:00	Constant XSEC Solutions
	12:00-13:30	Lunch break
	13:30-15:00	Multigroup Solutions
	15:00-15:15	Coffee break
	15:15-17:00	Infinite Medium Benchmarks (1)
	17:15-18:30	Special Seminar on Analytical Diffusion Theory
Thursday 27	9:00- 10:15	Infinite Medium Benchmarks (2)
	10:15-10:30	Coffee break
	10:30-12:00	Semi-infinite Medium Solutions
	12:00-13:30	Lunch break
	13:30-15:00	1D plane Solution: Response Matrix
	15:00-15:15	Coffee break
	15:15-17:00	1D plane Solution: Matrix Ricatti
	18:00-20:00	Thanksgiving Banquet
Friday 28	9:00- 10:15	1D plane Solution: Lagrange Interpolation
	10:15-10:30	Coffee break
	10:30-12:00	1D plane Solution: Adding and Doubling
	12:00-13:30	Lunch break
	13:30-15:00	1D Kinetic Theory
	15:00-15:15	Coffee break
	15:15-17:00	Participant's Choice
	17:00-17:30	Certificate of Completion Ceremony

*Tentative as of 2/9/25

A SHORT COURSE
on
ONE-DIMENSIONAL ANALYTICAL METHODS FOR
VERIFICATION OF
NEUTRON TRANSPORT ALGORITHMS



OECD/NEA
Paris, France
24-28 November 2025

Instructor: B.D. Ganapol
Professor
Department of Aerospace and Mechanical Engineering
University of Arizona
Ganapol@cowboy.arizona.edu

Research Professor
Department of Nuclear Engineering
University of Tennessee

Course Text: *Analytical Benchmarks for Nuclear Engineering Applications*, B. Ganapol, OECD © 2008, NEA No. 6292, Organisation for Economic Co-operation and Development

Reference Texts: *Nuclear Reactor Theory*, G. Bell and S. Glasstone, Van Norstrand and Reinhold, NY, 1970.

Computational Methods of Neutron Transport, E.E. Lewis and W.F. Miller, Jr, ANS Publishers 1993.

COURSE MOTIVATION

The study of the neutron transport equation is a subtle blend of theoretical mathematics, numerical methods and computational strategies addressing the interaction of neutrons and nuclei. Not only do we gain valuable physical insight from solutions of the transport equation, but we also create new mathematics, numerical methods and computational tools. Neutron transport theory, regarded as a branch of applied mathematics, plays a significant role in the design and analysis of nuclear systems. For instance, highly precise numerical evaluation of analytical solutions to the transport equation can serve as standards, also called semi-analytical benchmarks, to provide verification of numerical methods and coding of comprehensive algorithms for reactor analysis. With the passage of time however and with increasing computational advances and the success of Monte Carlo probabilistic transport, analytical methods of neutron transport theory is fast becoming a theory of the past. Thus the motivation for an OECD course on analytical benchmarks of transport theory.

The solution of the transport equation has been considered by some of the most creative and accomplished scholars of our time, including S. Chandrasekhar, E. Wigner, G. Plazcek, K. Case, N. Weiner , I Kuščur and P. Zweifel to name only a few. During the 60's and into the 70's, there was hardly an issue of *Nuclear Science and Engineering* or *Annals of Nuclear Energy* that did not contain at least two articles on the subject. Now, such articles are relatively infrequent at best and an ever-decreasing number appear primarily in applied mathematics and physics journals. Understandably, we are in danger of losing a beautiful and elegant theory, which, along with introducing several (semi-) analytical benchmarks, is the primary reason for offering a course on analytical methods in neutron transport theory.

In this course, we primarily focus on transforming theoretical solution representations of the mostly the1D neutron transport equation into numerically useable forms to generate highly precise solutions. While we mostly consider idealized 1D theory, it does provide meaningful features of transport solutions useful in benchmarking neutron transport algorithms. As will be demonstrated, 1D solutions can also provide 2D and 3D solutions.

COURSE OBJECTIVES

The course main objective is to provide a basis for fundamental concepts of numerical evaluation of analytical solutions to a variety of neutron transport equations. We include recent theoretical as well as numerical advances in analytical benchmarking. Course attendees will become familiar with

+ The derivation of the 1D transport equation and its various forms
+ Several analytical solution representations
+ Various numerical solutions with convergence acceleration for high order precision
+ Semi-analytical benchmarking strategies

…. and more.

To facilitate the course objectives, specially designed exercises encourage participants to engage in the physics and numerical methods used in generating semi-analytical benchmarks. In addition, several lessons have GUIs illustrating the programs in the suite of benchmarks that accompany the text. In this way, participants become acquainted with the benchmark library as well as experience the satisfaction of solving several relevant problems of analytical transport theory.

APPLICATION OF COURSE KNOWLEDGE

The course material will find use in various forms. First, knowledge of analytical solutions increases one's awareness of what is available as standards. While analytical solutions to idealized 1D- neutron transport scenarios do not necessarily apply directly to operating nuclear systems, they do provide guidance regarding operation of portions of a system. However, the most widespread use of the course material will be for the generation of benchmarking standards to which one compares results from proposed or legacy numerical algorithms-- thus enabling operational testing of an algorithm as well as overall assessment of performance. To this end, as emphasized, the course includes several demonstrations of the analytical benchmarks as well as exercises in analytical benchmarking practice.

COURSE PREPARATION

Participants should be familiar with fundamental reactor physics and the operation of nuclear systems. In addition, familiarity with mathematics from ODES through vector calculus and linear algebra is expected. The participant should also be familiar with elementary numerical methods and have some experience with computer programming primarily in MATLAB and FORTRAN. Finally, participants should come with an open mind awaiting new information.

COURSE CONTENT

We will consider neutron transport theory starting from the general to the specifics of 1D transport theory. The course will (tentatively) cover the following topics in the 1D and 2D benchmarks, time permitting:

The derivation of the neutron transport equation	**~ 3 Hrs**
Multigroup theory	**~ 2 Hrs**
Green's functions by Fourier Transform	**~ 4 Hrs**
Slowing down	**~ 2 Hrs**
Monoenergetic 1D transport theory*	**~ 8 Hrs**
2D Searchlight problem	**~ 3 Hrs**

***PROPOSED BENCHMARKS (for Benchmark Book Vol 2)**

1. The Fourier Transform Solution for the Green's Function of Monoenergetic Neutron Transport Theory

Nearly 60 years ago, Ken Case published his seminal paper on the singular eigenfunction solution for the Green's function of the monoenergetic neutron transport equation with isotropic scattering. Previously the solution came from Fourier transform. While it is apparent the two had to be equivalent, a convincing equivalence proof for general anisotropic scattering remained a challenge until now.

Ken Case, in a remarkably insightful paper-- one of the most cited papers in all of physics-- applied separation of variables to express the Green's function for monoenergetic, isotropically scattering neutrons in terms of singular eigenfunctions. Mika, shortly thereafter, considered anisotropic scattering. The isotropic scattering solution results from Fourier transform inversion. Showing equivalence between the singular eigenfunction and Fourier transform solutions for other than isotropic scattering is anything but straightforward. While equivalence holds, it was not intuitively obvious and the derivation lacked simplicity. Thus, the challenge of showing a straightforward equivalence of the two approaches remained. In the following, we revisit equivalence in a more unified manner through the Legendre polynomial expansion of the Fourier transformed solution in terms of analytically determined moments. In so doing, Chandrasekhar polynomials and generalized singular eigenfunctions naturally emerge to simplify the analytical transform inversion.

2. Matrix Riccati Equation Solution of the 1D Radiative Transfer Equation

While the 1D radiative transfer equation is probably the most solved transport equation with numerous numerical and analytical solutions available, few solutions can claim to be an extreme benchmark. In context, an extreme benchmark means high confidence in six to seven places (seven to eight significant figures) of the numerical solution for angular intensity. In addition, 1D means exclusively a homogeneous plane parallel medium possibly with a highly anisotropic scattering phase function represented either by a Legendre expansion or point-to-point discrete directions. The radiative transfer equation in question will simulate azimuthally integrated unpolarized radiation only. Thus, when referenced, the transport equation is for monochromatic- 1D- plane parallel- azithmuthally integrated- anistropically scattering- neutrons or unpolarized photons in a homogeneous medium. In addition, we limit our investigation to a perpendicular beam source impinging on the left (near) surface of a conservative (ω = 1) slab medium in vacuum. Our intent is to develop a radiative transfer (or neutron transport) benchmark to extreme precision for anisotropically scattering slabs over a range of modestly large optical thicknesses.

Our approach commonly falls under the category of the matrix-exponential method, which includes the discrete ordinates method or more generally the method of matrix operators and matrix Riccati equations. These methods all share discretization of particle directions. The solutions include eigenvalue/eigenvector expansion, spatial finite difference and iterative sweeping in particle directions as well as doubling and adding via the interaction principle.

3. A Mathematical Realization of Entropy through Neutron Slowing Down

Here, we solve the slowing down equation for elastic scattering of neutrons in an infinite homogeneous medium analytically by decomposing the neutron energy spectrum into collision intervals. Since scattering physically smooths energy distributions by redistributing neutron energy uniformly, it is informative to observe how mathematics accommodates smoothing of the scattering process, which increases entropy through disorder.

The neutron slowing down in an infinite homogeneous medium is a classic problem in neutron transport theory. Neutrons (test particles) collide elastically with nuclei (field particles) and thereby lose energy to nuclear recoil. Thus, we have a common collisional process as described by a balance

in energy phase space between a neutron source and neutrons scattering into and out of an infinitesimal energy increment leading to the slowing down equation. Without absorption, one can analytically solve this equation for the neutron collision density distribution as it tends toward its equilibrium state. In addition, neutron loss is possible through radiative capture but will not be considered. Limiting our investigation to an infinite medium has naturally eliminated spatial and directional variations. While the slowing down equation is deterministic, it nevertheless, describes a statistical scattering process as illustrated by the associated mathematics. In the following, we argue that the solution to the neutron slowing down equation characterizes evolution of disorder associated with neutron-nucleus collisions. While it is not strictly correct to attribute disorder to entropy, clearly starting from monoenergetic neutrons representing complete order, subsequent scattering creates disorder by uniformly redistributing neutron energy and recoil energy transfer to field particles.

4. The Response Matrix Discrete Ordinates Solution to the 1D Radiative Transfer (or the Neutron Transport) Equation

The discrete ordinates method (DOM) of solution to the 1D radiative transfer equation has been an effective method of solution for nearly 70 years. During that time, the method has experienced numerous improvements as numerical and computational techniques have become more powerful and efficient. Here, we again consider the analytical solution to the directionally discrete radiative transfer equation in a homogeneous medium by proposing a new, and consistent, form of solution that improves upon previous forms. Aided by a Wynn-epsilon convergence acceleration, its numerical evaluation can achieve extreme accuracy as demonstrated by comparison with published benchmarks. Finally, we readily extend the solution to a heterogeneous medium through the star product formulation producing a novel benchmark for closed form Henyey-Greenstein scattering as an example.

Every few years or so, someone comes up with a new numerical or analytical scheme to solve the radiative transfer equation (RTE) in 1D plane parallel geometry. This presentation is another one of those. The coherent scattering radiative transfer equation (RTE), best known for its application in investigations of solar radiation in planetary atmospheres, the Earth's atmosphere and oceans, and in the transport of neutrons in scattering, absorbing and fissionable media, is the focus. Its solution, in terms of discrete streams, gained prominence when initially suggested by Wick and subsequently applied to general anisotropic scattering media with polarization by Chandrasekhar. In the years following, many investigators contributed to the improvement of what we now know as the discrete-ordinates (numerical) method (DOM) of solution. Of particular note are the theoretical and applied contributions of many. Arguably, three individuals, K. Stamnes, C. E. Siewert along with R. M. Garcia particularly stand out for their contributions.

5. The LN/LN0 Method For 1D Neutron Transport In a Slab Medium

A new, highly precise benchmark for the monoenergetic 1D neutron transport equation with isotropic scattering, based on Case's singular eigenfunctions, is presented. Because of the nature of singular distributions, Case's analytical solution notoriously resists straightforward numerical computation. To overcome this difficulty, two complementary Lagrange interpolation schemes are constructed to achieve extreme precision (~8 to 10 places) for number of secondaries in the range $0.001 < c < 0.99999$ and slab thicknesses in the range 1 to 100.

Applying Case's method to express the analytical solution to any transport problem other than in an infinite medium requires half- range orthogonality and the accompanying complex variable analysis. While generating an analytical expression, its numerical evaluation generally involves solving integral equations. As a result, numerical evaluation is complicated because of the necessity, for example, to solve coupled non-linear integral equations to determine the X-Y functions for a slab, or Chandrasekhar's H-function for a semi-infinite medium. For this reason, a

numerical solution represents a challenge, even though it is for one-dimensional transport. Therefore, to avoid half– range orthogonality, Siewert ingeniously devised the FN method, based on full-range orthogonality, to construct a spectral expansion of the solution through collocation and analytical evaluation of the resulting integrals over singular eigenfunctions. Here, we continue in a similar, though novel, manner with Lagrange interpolation.

6. Response Matrix/Discrete Ordinates Solution of the 1D Fokker-Planck Equation

The Fokker-Planck equation (FPE) is one of the quintessential equations of particle transport theory. Representing small angle scattering characteristics of electron and photon transport by differential scattering indeed is a mathematical/numerical challenge. Here, we address the challenge with the method of response matrix applied to the SN approximation to arrive at a nearly six place precision benchmark. Our approach aligns with the response matrix solution of the radiative transfer equation for anisotropic scattering published previously. We conclude with the comparison of the benchmark to a classical finite difference approximation.

Appropriate characterization of particle scattering is crucial in any transport application. For neutrons, there is the scattering phase function, which generally is expressible as a Legendre polynomial expansion. For the FPE, say for electrons, the scattering representation is differential, but, like neutrons, has a matrix analog in the discrete ordinates sense. The common matrix formulations for both neutrons and electrons enables application of the Response Matrix/Discrete Ordinates method of solution RM/DOM previously applied to neutrons and photons. In extreme forward scattering however, such as required for detection of tumors by light or for screen-Rutherford scattering of electrons in cancer treatment, Fokker-Planck scattering has proven to be an effective scattering model. Thus, our focus in this work is the application of response matrix theory for application of the 1D FPE, which is new and numerically promising.

7. DPN Solution of the 1D Monoenergetic Neutron Transport Equation with Benchmarking

It is well known that expressing the angular flux to the neutron transport equation in a 1D-plane slab by the full-range PN approximation is problematic for two fundamental reasons. The first concerns how the equations close since there is always one more unknown than equation. The second is how one best represents half-range boundary conditions by a full-range expansion, which is not possible in any exact sense. There have been several prescriptions on how to treat these important issues, but no definitive conclusion on what the best is has been forthcoming. The best approach to specifying boundary conditions is to avoid the difficulty altogether by resorting to the DPN moments approximation. Now, the moments over the incoming fluxes on the slab near and far boundaries are exact. Since the DPN approximation also has the advantage of being a coupled set of ODEs for flux moments, the ODEs are solved following well known techniques. Finally, one way to close is through convergence acceleration in *N*, the number of moments of the DPN approximation.

8. Adding and Doubling Solution to the 1D Fokker-Planck Equation

The 1D Fokker-Planck equation (FPE) plays a major role in the propagation of light in the universe. It specifically describes small angle scattering of photons (and electrons) as they travel in participating media. In particular, the differential scattering term representing the phase function scattering law enables the small angle scattering. This term also makes the FPE a challenge to solve in the discrete ordinate sense. Our approach utilizes adding and doubling, which first successfully solved the linear Boltzmann equation in the 1960s. With the help of Morel's discrete ordinate equivalence of the angular Laplacian, the FPE becomes similar to the discrete ordinates equation of linear transport theory. We then take advantage of the similarity through adding and doubling for its solution.

We begin with the formation of the discrete ordinate approximation to the FPE. The formulation includes the discretization of the angular Laplacian required to preserve the first and second moments of particle intensity. The first order form of the equation naturally leads to a matrix exponential solution for which we employ a Crank Nicolson numerical approximation. By an appropriate choice of exponential approximation, we find the response matrix, which gives the exiting angular intensity in terms of the incoming intensity. Finally, we evaluate the matrix multiplications to give our final numerical benchmark of exiting intensities for incoming intensities to which we compare to a response matrix benchmark.

9. Poiseuille Channel Flow by Adding and Doubling

The flow of a rarefied gas between two plates of a micro channel, commonly called Poiseuille flow, has been extensively studied in the BGK approximation. In particular, the Analytical Discrete Ordinates (ADO) method determines the velocity profile and flow rate efficiently and accurately. Here, we present an equally efficient and precise solution using the method of doubling, which however is arguably a numerically simpler and a more natural approach avoiding eigenvalues and eigenvectors. Highly precise benchmark results for Poiseuille flow are reported.

Poiseuille flow is one of, if not, the simplest of all RGD flows. It is molecular flow down a pipe or between plates. Here, the flow is in the z-direction in a transversely infinite channel of height d. The flowing molecules slip along the upper and lower channel plates to varying degrees. The accommodation coefficient α is a flow parameter indicating the strength of slip at the wall. The fundamental assumption for Poiseuille flow is a long wide channel with laminar flow down the channel as described by the BGK scattering model approximation. The challenge to Poiseuille flow is to determine the molecular distribution in the x- direction transverse to the flow and the corresponding flow rate.